Long range correlations in quantum gravity

Donald E. Neville*
Department of Physics, Temple University Philadelphia, Pennsylvania 19122
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Smolin has pointed out that the spin network formulation of quantum gravity will not necessarily possess the long range correlations needed for a proper classical limit; typically, the action of the scalar constraint is too local. Thiemann’s length operator is used to argue for a further restriction on the action of the scalar constraint: it should not introduce new edges of color unity into a spin network, but should rather change preexisting edges by ± one unit of color. Smolin has proposed a specific ansatz for a correlated scalar constraint. This ansatz does not introduce color unity edges, but the [scalar, scalar] commutator is shown to be anomalous. In general, it will be hard to avoid anomalies, once correlation is introduced into the constraint, but it is argued that the scalar constraint may not need to be anomaly-free when acting on the kinematic basis.

I. INTRODUCTION

The recently proposed spin network formulation of quantum gravity [1–3] is very appealing, because it is similar to Wilson loop approaches [4] which have been used very successfully in quantum chromodynamics. Wilson loops are inherently non-local, and even though it is possible to construct non-local objects within the traditional approach utilizing quantized local fields, the spin network approach has focused attention on the non-local structures which seem to be needed to satisfy the constraints of quantum gravity.

Although local fields are used in the initial construction of spin network states and operators, the final result does not involve local fields. The physical meaning of a given state is encoded in the SU(2) “colors” assigned to a network of edges (one-dimensional curves embedded in three-dimensional space) and in the way in which these edges are connected together at vertices where edges meet. The operators of the theory alter the edges and vertices in specified ways, without explicitly invoking local fields.

In a Dirac constrained quantization framework, the spatial diffeomorphism constraints are relatively easy to satisfy, using spin networks, because information encoded in the edge colors and vertex connectivities is not sensitive to the way the edge curves are coordinatized in the underlying three-dimensional spatial manifold. Also, research into spin networks has led to greatly increased understanding of how to regulate quantum operators [5–7]. With hindsight, this understanding could have been achieved within the framework of standard quantum field theories. (The key is to consider only operators which respect spatial diffeomorphism invariance. In the language of differential geometry, the key is to smear n-forms over n-surfaces [8].) However, this insight into regularization was not achieved in the traditional framework, and the new approach should be given the credit for stimulating a great deal of new thinking.

There are disadvantages to an approach based on spin networks, rather than local fields. Smolin [9] has pointed out that any approach which downplays the continuum and relies on discrete structures (spin networks, Regge calculus) will not automatically possess the long range correlated behavior needed for a satisfactory classical limit. In the Regge calculus approach, long-range behavior emerges from discrete dynamics only when Newton’s constant and the cosmological constant are tuned to critical values such that a correlation length diverges in the limit of small edge lengths [10].

In the spin network approach, one does not approach the continuum by varying a length. One recovers the classical limit, and the classical fields, by demanding that edge lengths be small compared to the length scale over which the underlying fields vary appreciably.

In this limit, consider the scalar constraint, which is the hardest constraint to satisfy and the hardest constraint to understand intuitively. The classical limit does not determine the spin network scalar constraint uniquely: the literature contains two different spin network operators [11–13] which reduce to the same Dirac scalar constraint in the limit of small edge lengths. In this situation, one falls back on simplicity as a criterion to choose among the possibilities, but as emphasized by Smolin, the spin network scalar constraints proposed up to now are too local in character to give rise to long range correlations. More precisely, the scalar constraint produces both local and non-local effects, but the latter do not give rise to long range correlations. Local effects: when the scalar constraint acts at a vertex, the constraint typically changes the color of each edge as well as the connectivity of the vertex, that is, the way in which the SU(2) quantum numbers of the edges are coupled together to form an SU(2) scalar. This action is completely local (confined to a single point, the vertex). Non-local effects: if the edges radiating from a given spin network vertex are visualized as a set of spokes radiating from a hub, then the constraint adds an edge of color unity which connects a pair of edges meeting at the vertex, so that (if the constraint acts several times) the wheel begins to look like a spider’s web. Thiemann calls these connecting edges of color unity “extraordinary edges” [11]. This action is non-local because the extraordinary edges are attached to the spokes at a finite distance from the vertex. The non-local action produces no correlation, because, if one...
moves out along an edge from the original vertex to a neighboring vertex, the colors and connectivity at the neighboring vertex have not changed: the neighboring vertex has no way of “knowing” that anything has happened at the original vertex. This situation is illustrated in Figs. 1 and 2, which show a portion of a spin network before and after the scalar constraint acts at the upper vertex. (The spin network shown in the figures is a bit special, in that all vertices are trivalent, with only three edges meeting at each vertex; this does not affect the argument.)

The scalar constraint has changed the colors at the vertex where it has acted (local action), and has altered the diagram at some distance from the vertex by adding an extraordinary edge (non-local action), but if one moves out along the left-hand edge starting from the top vertex and ending at the bottom left-hand vertex, the latter vertex has not been changed. The action is non-local, but nevertheless uncorrelated.

Assuming the scalar constraint is Hermitian, there will also be matrix elements which remove one extraordinary edge and a state in the kernel of the constraint presumably would be an infinite series of states having ever increasing numbers of such edges. However, no matter how many extraordinary edges the state contains, if A and B are two nearest neighbor vertices (connected by an edge), then whether or not the constraint is satisfied at vertex A depends entirely on how the edges are arranged at A; the arrangement at vertex B is irrelevant. This does not resemble classical gravity, where what is going on at the Sun is supposed to affect what is happening at the Earth. Presumably in order to introduce correlations one must make the scalar constraint less local in character.

What features of the scalar constraint should one change in order to make it correlated? In a spin network basis, the basic ingredients used to construct a scalar constraint are $E^a_A$ triad operators separated by holonomies. (The Thiemann scalar constraint contains also volume and extrinsic curvature operators, but these in turn are constructed from triads and holonomies. See, for example, papers by Borissov, De Pietri, and Rovelli [14,18].) There must also be a triad rule specifying which holonomies the triad “grasps” (functionally differentiates) within the spin network and a holonomy rule specifying how the holonomies within the operator are to be aligned along the edges of the spin network. One can change either the fundamental arrangement of triads and holonomies, or the triad rule, or the holonomy rule (or some combination of all of these).

* A posteriori it has turned out to be difficult to change the fundamental arrangement of triads and holonomies. This fundamental structure must have the correct behavior under spatial diffeomorphisms; otherwise the arbitrary coordinate charts introduced when regulating the constraint do not disappear at the end of the calculation. Classically, one can achieve the proper diffeomorphism behavior simply by dividing by appropriate powers of $\det E^a_A$. Quantum-mechanically, this determinant has too many zero eigenvalues and is difficult to invert. Thiemann’s version of the scalar constraint has both correct diffeomorphism behavior and operators which are well-defined quantum-mechanically, but his recipe is non-obvious, to say the least. Here I do not try to guess a new arrangement of triads and holonomies. Instead I consider a change in the holonomy rule, in part because this is easier, but in part also because an independent argument suggests that the holonomy rule needs changing. This argument is based on Thiemann’s recently introduced length operator [15].

The present paper has four parts. The first part, at the end of this Introduction, gives another argument, based on the Thiemann length operator, that the (non-local, but uncorrelated) spin network scalar constraints constructed up to now are not physically plausible because they dramatically distort length relationships within the spin network. The second part, in Sec. II of this paper, explores what seems to be the simplest way of introducing correlation without distorting lengths: expand a certain color unity loop occurring in the definition of the scalar constraint, so that the loop fills an entire triangle of the spin network, not just a portion of the triangle. (This in effect changes the holonomy rule.)

The third part of the paper, in Sec. III, considers a specific recipe for a correlated scalar constraint, one essentially identical to the recipe proposed by Smolin, and shows that this specific choice has a scalar-scalar commutator which is anomalous. The last part of the paper, in Sec. IV, discusses attempts to determine the scalar constraint by constructing a four-dimensional spin network formalism.

Note that the Thiemann, uncorrelated choice for the scalar constraint is anomaly-free, almost trivially so. The Thiemann scalar constraint $C$ acts on one vertex at a time, so that $C$ may be written as a linear sum of operators, one for each vertex $A, B, \ldots$ in the spin network:

$$C = C_A + C_B + \cdots.$$ (1)
In the scalar-scalar commutator $C$ occurs smeared by arbitrary scalar functions $M$ and $N$:

$$\left[ \int MC, \int NC \right] = \left[ M_A C_A + M_B C_B + \cdots, N_A C_A + N_B C_B + \cdots \right]$$

$$= (M_N - M_B N_A) [C_A, C_B] + \cdots.$$  (2)

$M_A$ is the value of the smearing function at vertex $A$, etc. The scalar-scalar commutator becomes a series of cross terms $[C_A, C_B]$ which vanish because the action of $C$ at vertices $A$ and $B$ is not correlated. (The scalar-scalar commutator must vanish, not just equal the spatial diffeomorphism constraint, because the commutator is acting on the spin network basis, which is in the kernel of the diffeomorphism constraint.) Once the constraint is made correlated, the vanishing of $[C_A, C_B]$ is no longer automatic. Requiring the constraint operator to be anomaly-free does not restrict an uncorrelated constraint, but should strongly determine any correlated constraint.

[This is a one-paragraph aside to take care of a possible objection to the form of the commutator given in Eq. (2).] It might seem that Eq. (2) is incorrect, because the first scalar constraint adds two new vertices to the spin network, and these new vertices are not included in the sum, Eq. (2). For example, suppose $C_A$ acts first (the $C_B C_A$ term in the commutator) and $A$ is the vertex at the top of Figs. 1 and 2. Then $C_A$ adds two new vertices, the two at the ends of the color unity line in Fig. 2. But the sum in Eq. (2) includes only the original vertices $A, B, \ldots$, not the two new vertices. $C_B$, the constraint which acts second, does not “see” the two new vertices, however. The three edges meeting at the new vertices are coplanar. Thiemann’s constraint annihilates coplanar vertices, because it contains a factor of $e^{ijk}$, where $i, j, k$ range over the components of unit vectors pointing along the three edges grasped by the scalar constraint. The sum in Eq. (2) is therefore correct as it stands. This is important because otherwise the new vertices would make the commutator anomalous.

Although I spend some time in Sec. III to show that the specific formula considered there is anomalous, it is not altogether clear to me that freedom from anomalies is a reasonable criterion to impose. In the classical theory, the commutator algebra must be anomaly-free, in order for the theory to possess full diffeomorphism invariance [16]. Intuitively, the various three-dimensional time slices can be “stacked” so as to form a manifold with four-dimensional, not just three-dimensional, diffeomorphism invariance. It might seem natural to demand an anomaly-free commutator in the spin network case, but the spin network states form a basis which is only kinematical, not physical. [Each member of a kinematical basis is annihilated by the Gauss SU(2) internal rotation constraints and the spatial diffeomorphism constraints, but is not necessarily annihilated by the scalar constraint. The physical basis is that subset of the kinematical basis which is also annihilated by the scalar constraint.] Since the kinematical states are not physical, in general, there are no observational consequences even if the commutator is anomalous. (For example, in a path integral, the unphysical states are excluded from the path integration by appropriate functional delta functions, so that the unphysical states do not even occur as virtual states.) Further, on the physical subset, where there might be observational consequences, the commutator $[C, C]$ is trivially non-anomalous, since each factor of $C$ separately annihilates the state. Additionally, the spin network representation in effect replaces the continuum with a one-dimensional subset of edges and vertices. Perhaps one should not be surprised if full diffeomorphism invariance becomes difficult to implement in this situation.

I give up the requirement of anomaly-free commutators with some reluctance, because the specific recipe for introducing correlations discussed here is not unique. (One alternative possibility is discussed in Sec. II.) It is desirable to impose freedom from anomalies, in order to determine the scalar constraint as fully as possible. If this requirement is not imposed, then at present it is not clear what physical requirement fully determines the scalar constraint.

Currently Thiemann’s expression for the scalar constraint seems to be the one most widely accepted [11]. The Thiemann form makes extensive use of the volume operator, because that operator is invariant under spatial diffeomorphisms and therefore is relatively easy to regulate. However, in other respects the Thiemann constraint is very hard to work with. It is the sum of an Euclidean constraint, which involves one volume operator, plus a “kinetic” term, which involves three volume operators, and the volume operator itself is quite complex. Accordingly, in Sec. II, when demonstrating the presence of anomalies, I will not try immediately to generalize Thiemann’s constraint to a more correlated version. Instead, I will work with a generalization of the Rovelli-Smolin constraint [12,13], essentially the generalization suggested by Smolin [9]. This means I will have to pretend that the analytic factor in the matrix element can be regularized in a diffeomorphism invariant way, whereas in fact no one knows how to regulate this factor. [By “analytic factor” I mean every factor in the matrix element except the group theoretic factor contributed by the SU(2) dependence of the vertex.] Nevertheless, the calculation with the Rovelli-Smolin form should not be misleading, since the Rovelli-Smolin and Thiemann forms share certain crucial elements in common (in particular, the presence of color unity loops in the expression for the constraint operator).

Once the Rovelli-Smolin form has been discussed in detail, in Sec. II, it is straightforward to show how to introduce correlations into the Thiemann form of the constraint. The discussion of anomalies in Sec. III also uses the Rovelli-
Smolin form, but again this should not be misleading, because the anomaly is generated by the group theoretic factor, not by the lack of diffeomorphism invariance in the regulator.

I now discuss Thiemann’s length operator briefly, then use it to argue that any scalar constraint operator which introduces color unity edges will significantly distort length relationships within the spin network. The Thiemann length operator is one of three operators recently proposed to measure geometrical properties (length, area, and volume) of a spin network [5,8,15,17,18]. All three operators are spatially diffeomorphically invariant, so that any non-invariant structures introduced to regulate these operators drop out of final results. I will assume that all three operators are consistent with Euclidean geometry and with each other. For example, suppose one measures the volume of a spin network tetrahedron using the volume operator, then measures the length of each side of the tetrahedron using the length operator. From Euclidean geometry, the volume is also given by a determinant constructed from the lengths. For consistency, the volume measured directly should equal the volume computed from the lengths. There is no particular reason to expect consistency, except for spin networks which approximate classical states by demanding that the geometric operators are consistent, when acting on these states.) For the argument which follows, I will assume that all lines in the spin network, except the extraordinary lines added by the scalar constraint, have color much greater than unity, so that the state can be assumed to be classical, the geometric operators are consistent, and it is reasonable to use intuition based on Euclidean geometry.

Further, I will assume that all vertices are trivalent (three edges meet at each vertex) or, at least, that the network contains a subset which is entirely trivalent, because the spectrum of the length operator has been computed for such vertices. The restriction to trivalent vertices is probably not essential. Figure 1 shows such a trivalent subset consisting of six edges. The labels a–f are the colors of the edges; all colors are assumed to be order n, n ≫ 1. Figure 2 shows the same subset after the scalar constraint has acted once at the upper vertex and inserted an extraordinary edge. In Fig. 2 and succeeding figures, the notation b’ stands for b ± 1; similarly c’ = c ± 1, etc.; the action of the scalar constraint changes Fig. 1 into a weighted sum over the various possibilities for b’ and c’. In Fig. 2, I have suppressed a summation over b’ and c’ as well as the weighting coefficients.

I have drawn the upper half of the triangle as squeezed to a much smaller area, because the length operator predicts this is what happens to the triangle: the color unity line is short, and is inserted near the midpoints of edges b and c, rather than near the vertex. The lines labeled b and b’ = b ± 1, for example, both have lengths of order \( l_p n \), the Planck length, while the color unity line has length of order \( l_p \sqrt{n} \), very short compared to all the other lines in the diagram. This is intuitively a very implausible result. In the classical limit one thinks of the scalar constraint as almost commuting with other operators, such as the length operator. This implies the scalar operator should produce only a very small fluctuation in the geometry of the state, typically changing lengths by order \( l_p \), and therefore areas by \( \Delta A/A = \text{order} 1/n \).

I now verify in detail the statements made above about the lengths of edges in Fig. 2. If an edge joins two trivalent vertices, then Thiemann has shown that the squared length of the edge is a sum of two contributions, one from each vertex.

For example, for edge b in Fig. 2,

\[ L^2(b) = \lambda^2(b; e, d) + \lambda^2(b; b \pm 1, 1), \]

where

\[
(4/l_p^2)\lambda^2(c; a, b) = \frac{2c + 1}{2c + 1/2}[(a + b + c + 1)(a + b - c)(-a + b + c + 1/2)(a - b + c + 1/2)]^{1/2}
+ \frac{2c}{2c + 1/2}[(a + b + c + 1/2)(a + b - c + 1/2)(-a + b + c)(a - b + c)]^{1/2}.
\]

Equation (4) is very complicated in general, but in the limit that a, b, and c are large,

\[
\lambda^2(c; a, b) \to l_p^2 A(a, b, c),
\]

where A is the area of the triangle with sides (a/2,b/2,c/2). In the limit that one of the edges a, b, or c is an extraordinary edge, Eq. (5) is not quantitatively accurate, but does give the correct order of magnitude of \( \lambda^2 \). Thus the contribution to \( L^2 \) from a vertex with all three edges of order \( n \gg 1 \) is order \( l_p^2 n^2 \), while the contribution to \( L^2 \) from a vertex with one extraordinary edge is order \( l_p^2 n \). In Fig. 2, \( L^2(b) \) is the sum of a contribution of order \( l_p^2 n^2 \) from the vertex at the lower end of b, plus a negligible contribution of order \( l_p^2 n \) from the vertex with the extraordinary edge at the upper end of b. Similarly for the edge \( b' = b + 1 \); therefore the two edges b and b’ in Fig. 2 both have lengths of order \( l_p n \), and the vertex with the extraordinary edge occurs at the approximate midpoint of the side. The extraordinary edge itself is much shorter, since both vertices contribute order \( l_p^2 n \) to \( L^2 \), therefore order \( l_p \sqrt{n} \) to L. This completes the demonstration that length relationships are strongly distorted when a color unity line is added to a spin network.
II. CORRELATIONS

From the discussion in the Introduction, any scalar constraint formula which introduces color unity edges into a diagram will distort lengths. Also, if the color of an edge is not modified along its entire length, the constraint will affect only the vertex at one end of the edge, and there will be no correlation. The obvious solution to both these problems is to push the color unity edge in Fig. 2 all the way to the bottom of the triangle, until it coincides with edge $d$, then recouple so that the color one edge disappears, and the bottom edge has color $d \pm 1$. This is the qualitative idea, expressed in graphic terms.

To become more quantitative, one must revert to the underlying local field theory, construct the operators and the states in terms of local fields, and then infer the corresponding spin network operators and states, for both the correlated and uncorrelated version of the scalar constraint operator. The earlier steps in this procedure have been discussed at length in the literature [1,2,8,19], and I will repeat those discussions only enough to recall a few key steps in the procedure. Also, I will not try to rederive the SU(2) recoupling factors which occur at some steps.

There is universal agreement as to the field theoretic meaning of the spin network state: a holonomy matrix $h$ is associated with each edge of color $c$ in a spin network:

$$h(\gamma) = P \exp \left[ i \int ds \gamma_a(s) A^a \right], \quad (6)$$

where $P$ denotes path ordering, $\gamma_a$ is the tangent vector to the edge, the integration is over the entire edge, the $A^a$ are the connections for the rotational [SU(2)] subgroup of the local Lorentz group, and the $S_a$ are the generators of SU(2) for the irreducible representation having color $c$. An explicit factor $i$ is needed because I take the generators $S_A$ to be Hermitian.

Associated with each vertex is an SU(2) $3J$ symbol which couples the (suppressed) $S_2$ indices on the $h$’s to a total spin zero. (There will be more than one $3J$ symbol if the vertex is higher than trivalent)

There is less agreement as to the field theoretic meaning of the operators, since the classical limit does not determine the operators uniquely. I work primarily with the Rovelli-Smolin prescription for the scalar constraint, which is

$$C_{RS} = - \text{Tr} \left[ E_A^a(x_1) \sigma_A h_{ab}(x_1, x_2) \right]$$

where $E_A^a$ is the densitized inverse triad, the momentum variable conjugate to $A_a^a$, and the two $h_{ab}$ together form a small closed loop in the $ab$ plane. Of course the small loop serves to point split what would otherwise be a poorly defined product of $E$ field operators, and the holonomies $h$ must be inserted to keep the construction SU(2) invariant, but in addition the loop holonomies supply a factor of $F_{ab}$ needed for the classical limit, $F$ the field strength constructed from the connection $A_a^a$. If $h_{ab}(x_1, x_2)$ is a very short segment of loop, so that most of the area of the loop is enclosed by $h_{ab}(x_1, x_2)$, then in the classical limit, where the fields are varying slowly over the area of the loop,

$$h_{ab}(x_1, x_2) \approx 1,$$

$$h_{ab}(x_1, x_2) = 1 + i F_{ab}^c \sigma_c \epsilon_{abc}(\text{area})/2,$$

$$C_{RS} = \tilde{E}_A^a E_B^b F_{ab} \epsilon_{abc}(\text{area}), \quad (8)$$

where (area) is the total area enclosed by the loop. Assuming all the loops $h_{ab}$ are given the same area, independent of ab, one can divide out the area factor and arrive at (almost) the usual classical scalar constraint. ($C_{RS}$ has the wrong density weight to yield a diffeomorphism invariant when integrated over $d^3x$. As mentioned earlier, this non-invariance leads to difficulties when the constraint is regulated.)

Now continuing to work in the field theoretic language, allow $C_{RS}$, Eq. (7), to act on the state. Since the $E$ are canonically conjugate to the $A$ fields, $E_A^a$ acts on (or “grasps”) each holonomy in the state like a functional derivative $\delta \delta A_a^a$, and brings down a group-theoretic factor of $S_A$ from the exponential of the holonomy. The $S_A$ is multiplied by $\sigma_A$ from the scalar constraint, times analytic factors which I ignore. Because the tensors $S_A$ and $\sigma_A$ each carry two suppressed matrix indices in addition to the color two index $A$, $S$ and $\sigma$ are essentially $3J$ symbols which couple two color $c$ lines to form a color two line (in the case of $S$) or two color unity lines to form a color two line (in the case of $\sigma$).

Finally, translate this action back into the language of spin networks: each grasp by an $E$ introduces two vertices into the spin network (the two $3J$ symbols corresponding to $S_A$ and $\sigma_A$), while the $h_{ab}(x_1, x_2)$ in the scalar constraint introduces a small loop of color unity. Figure 3 shows the spin network which results when $C_{RS}$ acts at the upper vertex of Fig. 1. Note $E(x_1)$ and $E(x_2)$ must grasp two different edges in Fig. 3, because the antisymmetry of the scalar constraint in the indices $ab$ destroys terms where the two $E$’s grasp the same edge. Also $x_1$ and $x_2$ are close together, which means the two grasps must occur close to a vertex, as shown in Fig. 3. In fact all five vertices at the top of figure occur at exactly the same location. They are drawn slightly separated for clarity, but there is no holonomy on any of the edges connecting any of the five vertices. These holonomy-free edges are drawn as dotted lines in the figure. (The small color unity
FIG. 4. Figure 3 after two recouplings.

line at the top of the diagram carries the holonomy $h_{ab}([x_1,x_3])$, which is approximately unity.) SU(2) recoupling theory may be used to rearrange the $3J$ symbols, therefore, until the diagram resembles Fig. 2. (I am still working with an uncorrelated constraint and have not yet constructed the correlated constraint.) To obtain Fig. 2, one recouples the four colors connected to the color two line on the left (colors 1,1,b,b), as well as the four colors connected to the color two line on the right (colors 1,1,c,c); the result is Fig. 4. The spin network of Fig. 4 should be multiplied by two $6J$ symbols from the recoupling of the $b$ and $c$ lines; I suppress these for the moment. The five vertices at the top of Fig. 4 are still at the same point, but one can in effect shift the two lowest of these vertices downward in space by shifting an holonomy onto the $b'$ and $c'$ lines: factor the two holonomies on the color unity and color $b$ lines; using the standard identity giving the behavior of the $3J$ symbol under rotations.

\begin{equation}
D_{pp'}^{b}((\omega)D_{qq'}^{1}(\omega)\left(\begin{array}{c} b \\ p \\ q \end{array}\right)\left(\begin{array}{c} b' \\ p' \\ q' \end{array}\right)
\end{equation}

slide one factor from each line upward to form an holonomy on the color $b'$ line; do the same for the color $c'$ line. In this way one shifts the two lowest vertices downward to the midpoints of the $b$ and $c$ lines. If now one recouples to remove the small triangle with sides ($b,c,1$), the result, the uncorrelated scalar constraint action, is the spin network of Fig. 2.

It is straightforward to modify the above procedure to produce a correlated constraint: do not factor the $b$ and unity holonomies; slide both holonomies entirely upward onto the $b'$ edge, so that the lower ($b,1,b'$) vertex moves all the way down the $b$ side, to the lower left vertex of the original triangle. Similarly, slide the $c$ holonomy up onto the $c'$ and unity edges. The result is shown in Fig. 5.

If one wishes, one can now remove the lower color unity line entirely from the diagram. Recouple the color $d$ and color unity edges at the $b$ end (or at the $c$ end; it does not matter) as shown in Fig. 6. Slide the holonomies from the color $d$ and color unity lines onto the color $d'=d\pm 1$ line; this in effect moves the right-hand $(d,1,d')$ vertex all the way to the right and replaces the original pair of holonomies by a single holonomy having color $d'=d\pm 1$. There will still be color unity lines in two small triangles at each end of the new color $d\pm 1$ line; these triangles can be removed by a further recoupling.

The correlated version of the constraint will have the same classical limit as the uncorrelated version, provided the second line of Eq. (8) continues to hold. That is, the fields must be slowly varying over the entire area of the triangle $bcd$, not just over the upper half of the triangle.

One must also ask about the (area) which formerly was merely an overall constant to be divided out of the scalar constraint, last line of Eq. (8). This (area) equals the area enclosed by the color unity loop, now the entire area of the triangle. This area is no longer necessarily infinitesimal and can vary from triangle to triangle.

One can continue to divide it out: compute the area of each triangle by using the Thiemann length operator to compute the length of the sides; then use standard trigonometry to compute the area from the lengths; then (write the scalar constraint as a sum of terms, one for each area, and) divide each term in the scalar constraint by the area. The resultant formula is very ugly. I will argue that worrying about this factor amounts to taking the Rovelli-Smolin example too seriously, since the Thiemann formula for the scalar constraint does not suffer from this problem. Moreover, the area factor probably should not be divided out, since the scalar constraint always occurs multiplied by $d^3x$. Writing $d^3x = d(area)d(length)$, one sees that a factor of area belongs in the constraint. The real problem is not how to remove the area, but how to include a length.

Clearly the Rovelli-Smolin constraint does not have a wonderful analytic factor, but it does have a group-theoretic factor which is relatively simple, yet sufficiently complex to be informative. Both the Thiemann and the Rovelli-Smolin constraints contain $E$ operators which “grasp” the state, introducing vertices with color two lines into the spin network, and both constraints contain an holonomy which introduces a color unity loop into a triangle of the spin network. Follow-
ing Smolin, I have introduced correlation by expanding the color unity loop to fill the entire triangle. The same procedure works for the Thiemann case. The Thiemann constraint is a sum of two Euclidean scalar constraint operators, plus a kinetic term. The Euclidean operators are

$$
C_{E1} = \text{const} \times e^{abc} \text{Tr} \{ (h_{bc} - h_{cb}) h_{a}^{-1} h_{a}^{-1} V \};
$$

$$
C_{E2} = \text{const} \times e^{abc} \text{Tr} \{ h_{a}^{-1} V (h_{bc} - h_{cb}) \}. \tag{10}
$$

$h_{bc}$ and $h_{cb}$ are color unity loops of opposite orientation lying in the $bc$ plane. (Orientation counts since the indices on $h$ are not traced over.) In the classical limit these loops produce the factor of field strength $F$ and hence are entirely analogous to the color unity loop of the Rovelli-Smolin example. The third holonomy, $h_{a}$, is a straight line segment parallel to external edge $a$; hence $h_{a}$ introduces no length distortions, since it does not join two sides. Consequently, there is no obvious need to move it, when constructing the correlated version of the constraint, and I leave it alone. [Its commutator with the volume operator $V$ is needed to turn the three-grasp operator $V$ into a two-grasp operator, essentially the $\tilde{E}_a^2 \tilde{E}_b^2$ factor in the classical limit, Eq. (8).]

I will not write down the kinetic operator, because it is very complex, but the important point is that there are no loops. It involves three commutators of the form $(h_{bc} - h_{cb}) h_{a}^{-1} h_{a}^{-1} V$.

III. ANOMALIES

In order to check for anomalies, one needs the group-theoretic factors which arise when the scalar constraint acts on a spin network. Fortunately, the same factors occur repeatedly. Consider the factors which arise when the scalar constraint acts on the spin network of Fig. 1 to produce Fig. 5. These factors are

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{An alternative recoupling of the $b$ line.}
\end{figure}
The first factors, $bc$, I will call "grasp" factors, because they arise when the $\tilde{E}$ operators in the scalar constraint initially grasp the $b$ and $c$ lines. One can think of the $b$ line as made up of $b$ parallel color unity lines, and the factor of $b$ arises from $b$ identical diagrams where the $\tilde{E}$ grasps each color unity line in turn and similarly for the factor $c$. The next two curly brackets in Eq. (11) are the $6J$ symbols which arise when the four edges $(b,b,1,1)$ and $(c,c,1,1)$ are recoupled. This is the recoupling which changes Fig. 3 into Fig. 4. I will call these $6J$ coupled. This is the recoupling which changes Fig. 3 into Fig. 4.

The $6J$ symbols arise when the four edges $(b,b,1,1)$ and $(c,c,1,1)$ are connected by $b$ and $c$ parallel color unity lines, and the factor of $b$ arises from $b$ identical diagrams where the $\tilde{E}$ grasps each color unity line in turn and similarly for the factor $c$. The product of the final $6J$ symbol, the $\theta$ function, and the $\Delta_a$ function may be denoted collectively the "triangle" factor, since these three factors arise when the small triangle with sides $(b,c,1)$ is removed from the upper vertex in Fig. 4, to yield the simpler upper vertex shown in Fig. 2 or 5. $\theta(b,c,a)$ is essentially the square of a $3J$ symbol, while $\Delta_a = (-1)^a(a+1)$.

Only one other group theoretic factor is needed, a "color zero recoupling" factor which arises when at a later step the two lines at the bottom of Fig. 5 are recoupled to give Fig. 6:

$$\Delta_{d \pm 1} / \theta(d,1,d \pm 1).$$

(12)

This may be called "color zero" recoupling, because the initial two parallel lines $d$ and 1 are not linked by any color. General formulas for the $6J$ and $\theta$ functions have been worked out by Kaufmann and Lins [20] and specific numerical values have been tabulated by De Pietri and Rovelli [18]. Physicists trained in traditional recoupling theory may be slightly puzzled by the $6J$ recoupling factors given in Eqs. (11) and (12): the "$3J$" and "$6J$" symbols used here possess the same symmetries as the traditional $3J$ and $6J$ symbols, but are normalized differently [20,18].

For the final spin network state of highest weight ($b \pm 1 = b + 1$, and similarly all other $\pm 1 = + 1$), the factors in Eqs. (11) and (12) are especially simple:

$$\left\{ \begin{array}{ll} b & b+1 \\ 1 & 1 \\ 2 & \end{array} \right\} = 1/2,$$

$$\left\{ \begin{array}{ll} b & b+1 \\ c+1 & c \\ a & 1 \end{array} \right\} \theta(b,c,a)/\Delta_a = 1,$$

$$\Delta_{d \pm 1} / \theta(d,1,d \pm 1) = 1.$$

(13)

The recoupling and triangle factors are just constants; only the grasp factors are functions of the color arguments. This simplicity suggests the following strategy for discovering anomalies. When the $[\text{scalar, scalar}] [C,C]$ commutator acts upon a spin network such as that of Fig. 8, the result is a linear combination of spin networks with modified edges $b$ 

FIG. 9. A final state such that all colors have increased.
smearing. Vertex smearing is the procedure used in Eqs. (1) and (2): multiply the constraint by the value of the smearing function (Lagrange multiplier) at the vertex which is grasped. In Eq. (2), for example, $M_A$ is the value of the smearing function at vertex $A$, and $C_A$ is the sum of all terms in the scalar constraint which grasp some pair of edges ending at $A$. Now that the small loop is extended over an entire triangle, however, perhaps a more appropriate procedure is to use area smearing: multiply the constraint by the average value of $M$ on the triangle, for instance by $M_{bcd} = (M_B + M_C + M_D)/3$, for a triangle with sides $bcd$ and vertices $BCD$. The commutator, Eq. (2), would be replaced by

$$[\int MC, \int NC] = (M_{bcd} N_{def} - M_{def} N_{bcd}) [C_{bcd}, C_{def}] + \ldots,$$

where $C_{bcd}$ is the sum of all terms in the scalar constraint which grasp a pair of edges in the triangle $bcd$. Since the functions $M$ and $N$ are arbitrary, the commutator must vanish term by term. Each $[C_A, C_B] = 0$ for vertex smearing, and each $[C_{bcd}, C_{def}] = 0$ for area smearing.

Vertex smearing is easier to check, since $C_A$ contains fewer terms than $C_{bcd}$; therefore check vertex smearing first. Let vertex $A$ be the topmost vertex in Fig. 7; let vertex $B$ be the leftmost vertex, where edges $bde$ meet. Since I want only those grasps which lead to the final state of Fig. 9, I need to consider only the term in $C_A$ which grasps sides $bc$, and only the term in $C_B$ which grasps sides $de$. The result is

$$[C_A, C_B](\text{Fig. 8}) = \cdots + (\text{Fig. 9}) \times [bde - (d+1)e]bc (1/2)^4 \times [bc + b(d+1)e + (d+1)e + f(d+1)]$$

$$\times \left[ ef + ed + fd \right] - \left[ ef + e(d+1) + f(d+1) \right] \times \left[ bc + bd + cd \right] (1/2)^4. \quad (16)$$

This expression is anomalous also.

As discussed in the Introduction, it is not entirely clear that imposing freedom from anomalies is a reasonable thing to do. However, one might wish to impose the requirement that the anomalous terms in the commutator be small, in the classical limit, because in the classical, continuum theory a very wide range of fields may be used to carry representations of the diffeomorphism group, including fields which do not satisfy the scalar constraint. Presumably anomalous terms are ‘‘small’’ if the commutator $[C, C]$ has small matrix elements compared to matrix elements of $C^2$, in the limit that all colors $a, b, c, \ldots$ are order $n$, $n \gg 1$. From Eq. (15) or (16), this requirement is satisfied, since

$$[C, C]/C^2 = O(1/n). \quad (17)$$

IV. FOUR-DIMENSIONAL APPROACHES AND FURTHER DISCUSSION

Reisenberger and Rovelli [21] have suggested that a kind of $3 + 1$ dimensional ‘‘crossing symmetry’’ could be used to fix some of the arbitrariness in the scalar constraint. They construct a proper time coordinate $T$ and propagate a spin network from proper time $0$ to $T$ using a path integral formalism. Each path is weighted by an exponential $\exp[-i\int dT^* \mathcal{H}]$, where $\mathcal{H}$ is the usual gravitational Hamiltonian, a sum of constraints. They expand the exponential in powers of $H$, and visualize the action of each power of $H$ on the spin network as follows. (The diagrams are drawn in 2 + 1 rather than 3 + 1 dimensional spacetime for ease of visualization, and the vertical direction is the proper time direction $T$.) Stack Fig. 2 vertically above Fig. 1. Figure 1 represents a portion of the spin network at $T=0$, before $H$ has acted; Fig. 2 represents the same spin network at proper time $T$ after $H$ has acted. Connect Figs. 1 and 2 by three approximately vertical lines. All three lines begin at the ‘‘a’’ vertex of Fig. 1. One line connects the ‘‘a’’ vertex of Fig. 1 to the ‘‘d’’ vertex of Fig. 2; the other two lines connect the ‘‘a’’ vertex of Fig. 1 to the two new vertices at the ends of the color unity line in Fig. 2. The action of the scalar constraint therefore introduces a tetrahedron into spacetime: the bottom vertex is the ‘‘a’’ vertex of Fig. 1; the three vertical edges are the world line of this vertex and the world lines of the two new vertices introduced by $H$; the top face is the spin network triangle with edges $(b', c', 1)$ in Fig. 2. For later reference note that the three vertical edges of this tetrahedron are qualitatively different from the three horizontal edges $(b', c', 1)$. The vertical edges are world lines of vertices; they are not colored; they are not edges of any spin network.

Any time slice (horizontal slice) through the middle of the tetrahedron will separate it into a ‘‘past,’’ containing one vertex, and a ‘‘future,’’ containing three vertices. Reisen-
berger and Rovelli call this a (1,3) transition. They argue that $3 + 1$ diffeomorphism invariance should allow one to rotate the tetrahedron in $3 + 1$ dimensional space or, equivalently, time-slice the tetrahedron at any angle. This is what they mean by “crossing symmetry.” In particular, consider a slice which would put two vertices in the past and two in the future: by crossing symmetry, $H$ should allow not only (1,3) transitions, but also (2,2) transitions. [In a (2,2) transition the number of vertices does not change, but the way the vertices are coupled does change.]

A (2,2) slice is qualitatively different from a (1,3) slice, however. The (1,3) time slice cuts only the vertical, world line edges of the tetrahedron; the (2,2) slice cuts both world line edges and spin network edges. In short, the tetrahedron is not really 4D symmetric: its sides are not all equivalent.

Markopoulou and Smolin [22] have proposed a four-dimensional spin network formalism in which the fundamental tetrahedrons are more fully symmetric, because all edges are colored. Markopoulou and Smolin do not use their formalism to determine the form of the scalar constraint. Indeed it would be contrary to their philosophy to do so. In their approach, the classical Einstein-Hilbert action emerges at the end of a long renormalization group calculation. One starts from a microscopic action (as yet unspecified, but presumably highly symmetric). If the initial action is in the right universality class, then the renormalization group calculation yields correlated behavior over macroscopic scales, as required by the classical theory.

Sections I–III of this paper discusses three-dimensional spin networks, but the results of those sections should carry over readily to the $3 + 1$ dimensional approaches just discussed. Presumably the initial microscopic Markopoulou-Smolin action should be chosen so as to affect more than one vertex at a time, insert no edges of color unity, and obey a crossing symmetry (following Reisenberger and Rovelli).

In order to obtain correlation and eliminate length distortions, I have required that the color unity loop added by the scalar constraint should be pushed outward from the vertex where the constraint initially acts, until the loop fills an entire triangle of the spin network. Consequently the constraint will not change the number of vertices in the spin network (although it can change the number of lines). For example, the uncorrelated scalar constraint action shown in Fig. 2 has added two new vertices, whereas the correlated action shown in Fig. 5 adds no new vertices (after recouplings which remove the color unity lines).

To see how the constraint could add a new edge, set $d = 0$ in Fig. 2 (no edge present initially). The two bottom vertices are now divalent, but that can be remedied, if desired, by adding new external lines; the valence will not matter. As before, let the constraint act at the top vertex, push the loop down the sides of the triangle, and recouple so as to fill the entire triangle (as in Figs. 3–5, now with $d = 0$). The $d = 0$ edge is replaced by $d' = 1$: a new edge has been added.

One could demand that the procedure be modified so that not even a new edge is added. For example, suppose the triangle with edges $bcd$ in Fig. 2 were part of the larger spin network shown in Fig. 8. Again set $d = 0$ and let the scalar constraint act at the upper vertex, but do not stop when the color unity loop has filled the upper half triangle of Fig. 8; continue to push this color unity loop until it fills the entire diamond. After recouplings such that $b \rightarrow b', e \rightarrow e', \ldots$, there is neither a new vertex nor a new edge.

There is some rationale for stopping when the loop has filled only the upper half of the diamond, however: if one thinks of the spin network as triangulating the space, then for pairs of edges which share a common vertex, such as the pair $(b, c)$ in Fig. 8, there should be a third edge which connects the ends of $b$ and $c$ away from the vertex, so as to form a triangle; otherwise the structure is not rigid, in general. One could say that the network of Fig. 8 with $d = 0$ really has a horizontal edge across the middle of the diamond; the edge just happens to have the special color value zero. Put another way, the scalar constraint should leave the number of vertices fixed, while adding enough edges to form a rigid structure. This idea is appealing in its simplicity, but sometimes simplicity does not equate to validity, as we have just seen in the case of the uncorrelated scalar constraint. Further thought is needed.

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