Conformal Geometry via Geometric Algebra

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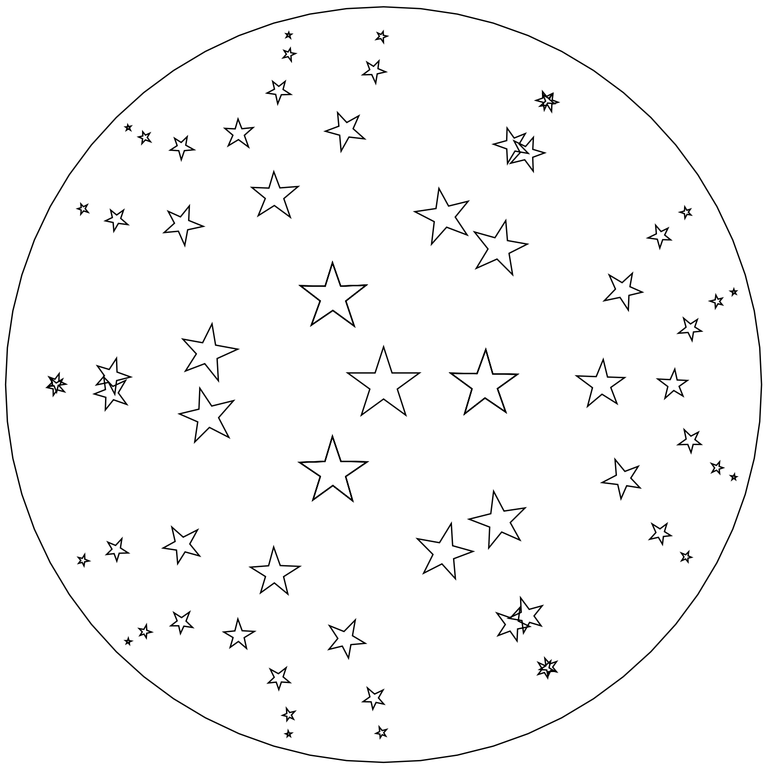
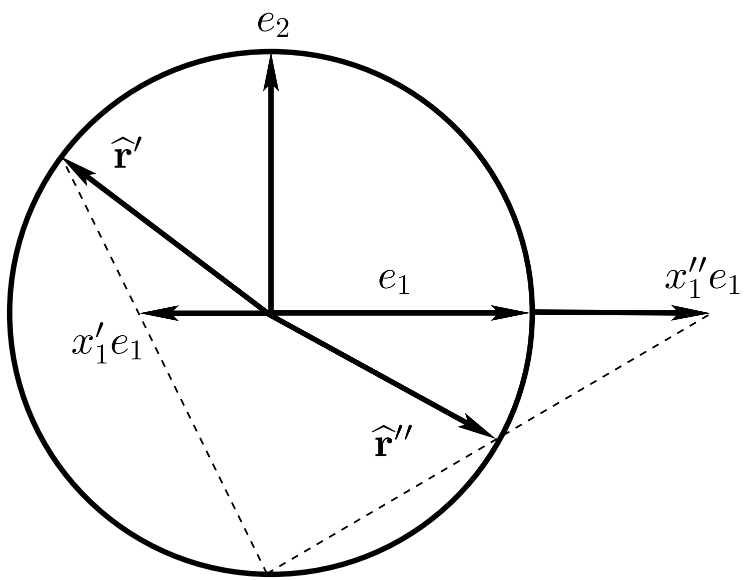
Hyperbolic Translation



Conformal Geometry

In Conformal Geometry the vector space $\mathcal{V}(p,q)$ is enlarged to $\mathcal{V}(p+1,q+1)$ with a mapping function $F : \mathcal{V}(p,q) \rightarrow \mathcal{V}(p+1,q+1)$ such that $F(x)^2 = 0 \; \forall x \in \mathcal{V}(p,q)$. $F(x)$ is a null vector. The motivation for this is that translations, rotations, dialations, and inversions in $\mathcal{V}(p,q)$ can all be encoded as rotations in $\mathcal{V}(p+1,q+1)$. The geometric algebra of $\mathcal{G}(p+1,q+1)$ makes handeling the rotations very simple. The starting point of conformal geometric is the stereographic projection of the line as show:



The points $x_1'e_1$ and $x_1''e_1$ are mapped into the unit vectors $\widehat{\mathbf{r}}'$ and $\widehat{\mathbf{r}}''$. A little algebra shows that if we let $x = x_1e_1$ be a general point on the line

$$\widehat{\mathbf{r}} = \frac{2x_1}{1+x_1^2}e_1 + \frac{1-x_1^2}{1+x_1^2}e_2 \tag{1}$$

But this representation is not homogenous since $\widehat{\mathbf{r}}^2 \neq 0$. Rename e_2 to $-e$[1] to distinguish it from the basis vector(s) of the space being described (in this case the line with basis e_1) and scale and add the vector $\bar{e}$ to equation 1 to get

$$X = 2x_1e_1 - \left(1 - x_1^2\right)e + \left(1 + x_1^2\right)\bar{e} \tag{2}$$

where

$$e^2 = 1, \qquad \bar{e}^2 = -1, \qquad e \cdot \bar{e} = 0$$

[1] The sign is reversed so that our final result is the same as Hestenes.

and $X^2 = 0$ and the vector space $\mathcal{V}(1,0)$ has been extended to $\mathcal{V}(2,1)$. To put equation 2 into standard form let

$$\begin{gathered} n = e + \bar{e}, \qquad \bar{n} = e - \bar{e} \\ n^2 = \bar{n}^2 = 0, \qquad n \cdot \bar{n} = 2 \\ X = 2x_1 e_1 + x_1^2 n - \bar{n} \end{gathered}$$

Then the general mapping $F : \mathcal{V}(p,q) \rightarrow \mathcal{V}(p+1, q+1)$ for $x \in \mathcal{V}(p,q)$ is given by

$$F(x) = \alpha \left(x^2 n + 2x - \bar{n} \right) \tag{3}$$

This is referred to as the Hestenes mapping since it was discovered by David Hestenes. We scale X by α to indicate that any multiple of X is also fulfills the $F(x)^2 = 0$ requirement. We then set α to make it simple to recover x from $F(x)$ by imposing the arbitrary normalization condition

$$F(x) \cdot n = -1 \tag{4}$$

which gives $\alpha = \frac{1}{2}$ and

$$F(x) = \frac{1}{2}\left(x^2 n + 2x - \bar{n}\right), \ \forall x \in \mathcal{V}(p,q) \tag{5}$$

If x is a dimensional quantity we introduce the scalar λ with the same dimensions and define

$$F(x) = \frac{1}{2\lambda^2}\left(x^2 n + 2\lambda x - \lambda^2 \bar{n}\right) \tag{6}$$

Note that for simplicity let $\lambda = 1$ and note that $n \cdot x = \bar{n} \cdot x = 0$ giving

$$\begin{aligned} F(x) \cdot F(y) &= \frac{1}{4}\left(x^2 n + 2x - \bar{n}\right) \cdot \left(y^2 n + 2y - \bar{n}\right) \\ &= -\frac{x^2}{2} - \frac{y^2}{2} + x \cdot y \end{aligned}$$

$$= -\frac{1}{2}(x-y)^2 \qquad (7)$$

so that the inner product in $\mathcal{V}(p+1, q+1)$ encodes the distance between two points in $\mathcal{V}(p, q)$. Note that $F(x) \cdot F(y) = 0$ implies that $x = y$, such are the wonders of null vectors.

Conformal Transformations

Conformal transformations consist of translations, rotatations, dilations, and inversions in $\mathcal{V}(p,q)$. In the conformal space $\mathcal{V}(p+1,q+1)$ all these transformations can be represented by rotations and reflections.

1. Rotations in the base vectorspace $\mathcal{V}(p,q)$

   If R is a rotation in $\mathcal{V}(p,q)$ then $RnR^\dagger = n$ and $R\bar{n}R^\dagger = \bar{n}$ since n and $\bar{n}$ commute with R and

$$
\begin{aligned}
RF(x)R^\dagger &= \frac{1}{2}R\left(x^2n + 2x - \bar{n}\right)R^\dagger && (8)\\
&= \frac{1}{2}\left(x^2RnR^\dagger + 2RxR^\dagger - R\bar{n}R^\dagger\right) && (9)\\
&= \frac{1}{2}\left(x^2n + 2RxR^\dagger - \bar{n}\right) && (10)\\
&= F\left(RxR^\dagger\right) && (11)
\end{aligned}
$$

2. Dilations in the base vectorspace $\mathcal{V}(p,q)$

Define the rotor $D_\alpha = \mathrm{e}^{\left(\frac{\alpha e\bar{e}}{2}\right)}$ in $\mathcal{V}(p+1,q+1)$. Since $(e\bar{e})^2 = 1$

$$D_\alpha = \cosh\left(\frac{\alpha}{2}\right) + \sinh\left(\frac{\alpha}{2}\right) e\bar{e} \tag{12}$$

and

$$D_\alpha n D_\alpha^\dagger = \mathrm{e}^{-\alpha} n \quad D_\alpha x D_\alpha^\dagger = x \quad D_\alpha \bar{n} D_\alpha^\dagger = \mathrm{e}^{\alpha} \bar{n} \tag{13}$$

so that

$$D_\alpha F(x) D_\alpha^\dagger = \frac{1}{2}\left(\mathrm{e}^{-2\alpha} x^2 n + 2\mathrm{e}^{-\alpha} x - \bar{n}\right) \tag{14}$$

when $F(x)$ is properly normalized and thus x has been scaled by $\mathrm{e}^{-\alpha}$.

3. Translations in the base vectorspace $\mathcal{V}(p,q)$

Define the rotor $T_a = \mathrm{e}^{\frac{na}{2}}$ and note that $(na)^2 = 0$ so that

$$T_a = 1 + \frac{na}{2} \tag{15}$$

and

$$\begin{aligned} T_a n T_a^\dagger &= \left(1+\frac{na}{2}\right) n \left(1-\frac{na}{2}\right) &(16)\\ &= n &(17)\\ T_a \bar{n} T_a^\dagger &= \left(1+\frac{na}{2}\right) \bar{n} \left(1-\frac{na}{2}\right) &(18)\\ &= \bar{n} - a^2 n - 2a &(19)\\ T_a x T_a^\dagger &= \left(1+\frac{na}{2}\right) x \left(1-\frac{na}{2}\right) &(20)\\ &= x + (a\cdot x)\, n &(21)\end{aligned}$$

so that

$$T_a F(x) T_a^\dagger = \frac{1}{2}\left(\left(x^2 + 2a \cdot x + a^2\right) n + 2(x+a) - \bar{n}\right) \tag{22}$$

and $x \to x + a$.

4. SpecialTransformation in the base vectorspace $\mathcal{V}(p,q)$

Define the rotor $\bar{T}_a = \mathrm{e}^{\dfrac{\bar{n}a}{2}}$ and note that $(\bar{n}a)^2 = 0$ so that

$$\bar{T}_a = 1 + \frac{\bar{n}a}{2} \tag{23}$$

and

$$\bar{T}_a n \bar{T}_a^\dagger = \left(1 + \frac{\bar{n}a}{2}\right) n \left(1 - \frac{\bar{n}a}{2}\right) \tag{24}$$

$$
\begin{aligned}
&= n - a^2\bar{n} - 2a && (25) \\
\bar{T}_a\bar{n}\bar{T}_a^\dagger &= \left(1 + \frac{\bar{n}a}{2}\right)\bar{n}\left(1 - \frac{\bar{n}a}{2}\right) && (26) \\
&= \bar{n} && (27) \\
\bar{T}_a x\bar{T}_a^\dagger &= \left(1 + \frac{\bar{n}a}{2}\right)x\left(1 - \frac{\bar{n}a}{2}\right) && (28) \\
&= x + (a \cdot x)\,\bar{n} && (29)
\end{aligned}
$$

so That

$$
\bar{T}_a F(x)\,\bar{T}_a^\dagger = \frac{1}{2}\left(x^2 n + 2\left(x + x^2 a\right) - \left(1 + 2a \cdot x + a^2 x^2\right)\bar{n}\right) \quad (30)
$$

and $x \to \left(x^{-1} + a\right)^{-1}$.

5. Inversions in $\mathcal{V}(p,q)$

$$ene = e(e+\bar{e})e = e - \bar{e} = \bar{n} \tag{31}$$

$$e\bar{n}e = e(e-\bar{e})e = e + \bar{e} = n \tag{32}$$

$$exe = -x \tag{33}$$

Evaluating $eF(x)e$ and normalizing

$$eF(x)e = \frac{1}{2}\left(x^2\bar{n} - 2x - n\right) \tag{34}$$

$$-\frac{1}{x^2}(eF(x)e) = \frac{1}{2}\left(\frac{1}{x^2}n - 2\frac{x}{x^2} - \bar{n}\right) \tag{35}$$

and $x \to x^{-1}$

Composition of transformations is as expected. If $T(a) = e^{\frac{na}{2}}$, $R(B, \phi) = e^{\frac{B\phi}{2}}$, and $D(\alpha) = e^{\frac{\alpha e\bar{e}}{2}}$ and we wish to translate, rotate, dilate, and invert (in that order) the composite transformation f is

$$f = eD(\alpha) R(B, \phi) T(a) \tag{36}$$

and

$$F(x') = fF(x) f^{\dagger} \tag{37}$$

where

$$f^{\dagger} = T(a)^{\dagger} R(B, \phi)^{\dagger} D(\alpha)^{\dagger} e^{\dagger}, \text{ Note that } e^{\dagger} = e \tag{38}$$

Geometric Primitives

- Blade Geometry

  Let $L = P_1 \wedge \ldots \wedge P_r$ be an r-blade such that $L \neq 0$ where the P_l are $P_l = F(p_l)$ and the $p_l \in \mathcal{V}(p,q)$. Then $P_l^2 = 0$ and the solutions of

$$L \wedge X = 0 \tag{39}$$

  where $X = F(x)$ and $x \in \mathcal{V}(p,q)$ will be shown to be primitive geometric shapes (points, lines, circles, etc.). In general the solution of equation 39 will be a linear combination of the P_l so that

$$X = \sum_l \alpha_l P_l. \tag{40}$$

  Also note that $L \wedge P_l = 0$ so that the p_l must be points in the geometric primitive.

Finally consider the effect of the conformal transformation R on equation 39. We have (using equation 43 in introGA.pdf)

$$R\left(L \wedge X\right)R^{\dagger} = \left(RLR^{\dagger}\right) \wedge \left(RXR^{\dagger}\right) = 0 \quad (41)$$

since $\left(RXR^{\dagger}\right)^2 = 0$ the blade $L' = \left(RLR^{\dagger}\right)$ represents a transformed geometric entity.

Additionally if $L = P_1 \wedge \ldots \wedge P_r$ then

$$RLR^{\dagger} = RP_1R^{\dagger} \wedge \ldots \wedge RP_rR^{\dagger} \quad (42)$$

and the points that define L are tranformed in exactly the same way as the point X by R.

- Subspaces:

  In the conformal space $\mathcal{V}\left(p+1, q+1\right)$ each blade defines a subspace such that the defining blade is an unnormalized pseudo-scalar for that subspace. For example consider the line $L = X_1 \wedge X_2 \wedge n$. It is contained in the subspace defined by the plane $P = X_1 \wedge X_2 \wedge n \wedge \bar{n}$ $(F(0) = \bar{n})$ since P is just a plane through the origin containing the line L. This is very useful when we wish to construct dual vectors to any grade blade. If we did not use this we could not construct vectors dual to lines and circles.

- Points, Lines, Circles, Planes, and Spheres:

  Let four points in 3-space be defined as $a = e_0$, $b = e_1$, $c = -e_0$ and $d = e_2$ with conformal maps $A = F(a)$, $B = F(b)$, $C = F(c)$, and $D = F(c)$ and consider the blade geometry functions.

  - Points:
    If $L = P_1 \wedge P_2$ where $P_1 \neq P_2$ $(p_1 \neq p_2)$ then the solutions are of the form

$$X = \alpha_1 P_1 + \alpha_2 P_2 \tag{43}$$

but since

$$X^2 = 2\alpha_1\alpha_2 P_1 \cdot P_2 = 0, \tag{44}$$

but $P_1 \cdot P_2 = -2\left(p_2 - p_1\right)^2 \neq 0$ so that the possible solutions are $\alpha_1 \neq 0$ and $\alpha_2 = 0$ or $\alpha_2 \neq 0$ and $\alpha_1 = 0$ so that $X = P_1$ or $X = P_2$ (scale factors do not matter since P's are null vectors) and L encodes the points p_1 and p_2.

Also note that (using equation 10 of introGA.pdf)

$$L^2 = \left(P_1 \cdot P_2\right)^2 \tag{45}$$

so that if $L^2 < 0$ it cannot be the outer product of two null vectors. This can happen when you calculate the intersection of some objects and indicates that there is no intersection.

If $L^2 \geq 0$ then P_1 and P_2 can be extracted by construction. Let a be a vector such that $a \cdot L \neq 0$, assume P_1 and P_2 are normalized such that $P_1 \cdot P_2 = -1 \rightarrow L^2 = 1$ and constuct the following vectors:

$$a' = a - (a \wedge L)\, L = -\left(\left(P_2 \cdot a\right) P_1 + \left(P_1 \cdot a\right) P_2\right) \tag{46}$$

$$A^{\pm} = a' \pm a' L = \left\{ \begin{array}{c} -2\left(P_2 \cdot a\right) P_1 \\ -2\left(P_1 \cdot a\right) P_2 \end{array} \right\} \text{, and } \left(A^{\pm}\right)^2 = 0 \tag{47}$$

and we have recovered P_1 and P_2 to within a scale factor.

Consider the following cases where $x = x_0 e_0 + x_1 e_1 + x_2 e_2$.

– Lines:
Caculate the blade (`GAsympy.py` was used for all four cases)

$$A \wedge B \wedge n \wedge X = \begin{array}{l} -2x_2 e_0 \wedge e_1 \wedge e_2 \wedge n \\ +\left(x_1 + x_0 - 1\right) e_0 \wedge e_1 \wedge n \wedge \bar{n} \\ +x_2 e_0 \wedge e_2 \wedge n \wedge \bar{n} \\ -x_2 e_1 \wedge e_2 \wedge n \wedge \bar{n} \end{array} \tag{48}$$

If it set to 0 the scalar equations represented are

$$x_2 = 0 \tag{49}$$

$$x_1 + x_0 = 1 \tag{50}$$

– Circles:

$$A \wedge B \wedge C \wedge X = \begin{array}{l} -x_2 e_0 \wedge e_1 \wedge e_2 \wedge n \\ +x_2 e_0 \wedge e_1 \wedge e_2 \wedge \bar{n} \\ +\frac{1}{2}\left(x_0^2 + x_2^2 + x_1^2 - 1\right) e_0 \wedge e_1 \wedge n \wedge \bar{n} \end{array} \tag{51}$$

If it set to 0 the scalar equations represented are

$$x_2 = 0 \tag{52}$$

$$x_0^2 + x_1^2 + x_2^2 = 1 \tag{53}$$

– Planes:

$$A \wedge B \wedge n \wedge D \wedge X = (x_0 + x_1 + x_2 - 1)\, e_0 \wedge e_1 \wedge e_2 \wedge n \wedge \bar{n} \quad (54)$$

If it set to 0 the scalar equations represented are

$$x_0 + x_1 + x_2 = 1 \quad (55)$$

– Spheres:

$$A \wedge B \wedge C \wedge D \wedge X = \left(x_0^2 + x_1^2 + x_2^2 - 1\right) e_0 \wedge e_1 \wedge e_2 \wedge n \wedge \bar{n} \quad (56)$$

If it set to 0 the scalar equations represented are

$$x_0^2 + x_1^2 + x_2^2 = 1 \quad (57)$$

- Transformation of Geometric Shapes:

In all four cases the relevant blade represents the correct geometric element through the correct points. By using the operations of translation, dialation, and rotation (all represented by rotations in conformal space) can be transformed into the general case for each geometric element.

- Alternate Construction of Circle:

If $A \wedge B \wedge C \wedge X = 0$ defines a circle it is easy to construct a circle through two points with a tangent vector given at one of the two points. Let the points on the circle be A and B and let the tangent vector be u. Then conside $C = F(b + \alpha u)$ so that the circle is defined by $F(a) \wedge F(b) \wedge F(b + \alpha u)$, but

$$F(b + \alpha u) = F(b) + \alpha\left((b \cdot u)\, n + u\right) + \frac{\alpha^2 u^2}{2} n \qquad (58)$$

To first order in α the blade defining the circle is then

$$F(a)\wedge F(b)\wedge(F(b)+\alpha((b\cdot u)n+u)) = \alpha F(a)\wedge F(b)\wedge((b\cdot u)n+u) \tag{59}$$

Since $F(a)$ and $F(b)$ are homogenous α can be made very small and still be absorbed into either so that the blade representing the circle throug[h] a and b with tangent u at b is

$$F(a)\wedge F(b)\wedge((b\cdot u)n+u) \tag{60}$$

Extraction of Geometric Shape Parameters from Blades (Dual Representations)

The duality operation for the $\wedge$ and $\cdot$ products is

$$X \cdot (A_r I) = (X \wedge A_r)\, I \tag{61}$$

where A_r is a pure r grade multivector and I is an appropriate psuedo-scalar. In our case A_r is a grade r blade representation of a geometric entity and I is a blade of one grade higher that contains A_r. We now consider how to extract geometric information from different blade representations.

- Lines and Planes:

  Both lines and planes can be described by the following equation (where x and x_0 are **vectors** restricted to their respective subspaces):

$$(x - x_0) \cdot x_0 = x \cdot x_0 - x_0^2 = 0 \tag{62}$$

  where x is a vector from the origin to the line or plane and x_0 is vector from the origin to the closest point on the line or plane. If the blade P represents a plane its dual vector is

$$P^* = PI = p + \alpha n \tag{63}$$

  where I is the psuedo-scalar for the entire conformal space. If L represents a line then its dual vector is

$$L^* = L\,(L \wedge \bar{n}) = l + \beta n \tag{64}$$

where $L \wedge \bar{n}$ is a plane containing L and the origin. Using the dual relationship we have that

$$\begin{array}{c} X \wedge P = 0 \rightarrow X \cdot P^{*} = 0 \\ X \wedge L = 0 \rightarrow X \cdot L^{*} = 0 \end{array} \tag{65}$$

For the case of the plane (or line) we have

$$F(x) \cdot (p + \alpha n) = \frac{1}{2}\left(x^{2} n + 2x - \bar{n}\right) \cdot (p + \alpha n) \tag{66}$$

$$= x \cdot p - \frac{1}{2}\alpha n \cdot \bar{n} \tag{67}$$

$$= x \cdot p - \alpha \tag{68}$$

Normalizing gives

$$x \cdot \frac{p}{\alpha} - 1 = 0 \tag{69}$$

or

$$\frac{p}{\alpha} = \frac{x_0}{x_0^2} \tag{70}$$

Note that if the plane or line contains the origin $\alpha = 0$ or $\beta = 0$ and p or l is a normal to the plane or line. x_0 completly determines the plane. The line is completely determined by x_0 and

$$\Delta p = p_1 - p_2 = \frac{1}{2}(L \cdot n) \cdot \bar{n} \tag{71}$$

where p_1 and p_2 are the points on the line encoded by L. Thus Δp is a vector in the direction of the line and x_0 and Δp determine the line.

- Circles and Spheres: Both circles and spheres are described by the equation

$$(x - x_0)^2 = \rho^2 \tag{72}$$

If $C = X_1 \wedge X_2 \wedge X_3$ is a circle and $S = X_1 \wedge X_2 \wedge X_3 \wedge X_4$ is a sphere the dual vectors can be constructed as follows

$$\begin{aligned} C^* &= C\,(C \wedge n) \\ S^* &= S\,(S \wedge n) \end{aligned} \tag{73}$$

Note that $C \wedge n$ is a plane that contains C and $S \wedge n$ is a hyper-plane that contains S so that each blade can respectively act as a psuedo-scalar for the circle and sphere.

If C^* and S^* have a non-zero coefficent for $\bar{n}$ they can be put in the form (with proper scaling)

$$C^* \text{ or } S^* = F\,(x_0) - \frac{\rho^2}{2}n \tag{74}$$

but by duality we have $X \cdot C^* = 0$ and $X \cdot S^* = 0$ so that

$$F\,(x) \cdot \left(F\,(x_0) - \frac{\rho^2}{2}n\right) \quad = \quad 0 \tag{75}$$

$$(x - x_0)^2 - \rho^2 = 0 \tag{76}$$

Thus S^* and C^* gives the center and radius of both the sphere and circle. The orientation of the circle is given by

$$\Delta p_2 \wedge \Delta p_3 = (p_2 - p_1) \wedge (p_3 - p_1) = \frac{1}{2}((C \wedge n) \cdot n) \cdot \bar{n} \tag{77}$$

where p_1, p_2, and p_3 are the points that define the blade C. Since this result is a grade two blade all the formulas hold in higher dimension spaces.

Intersections of Geometric Objects

If two geometric entities are represented by the dual vectors B_1^* and B_2^* then the intersection of these entities are the set of points, X (rays in the conformal space) that simultaneously satisify

$$X \cdot B_1^* = X \cdot B_2^* = 0 \tag{78}$$

These two conditions are equivalent to the single condition

$$X \cdot (B_1^* \wedge B_2^*) = 0 \tag{79}$$

since

$$X \cdot (B_1^* \wedge B_2^*) = (X \cdot B_1^*)\, B_2^* - (X \cdot B_2^*)\, B_1^* \tag{80}$$

and

$$X \wedge (B_1^* \wedge B_2^*)^* = 0 \tag{81}$$

Where the dual of the outer product is calculated using the minimal psuedo-scalar that contains the intersection. For example if B_2 and B_2 are some combination of planes and spheres the intersection will be circles, a point, or no intersection at all. To simplify for now assume we are dealing with entities in $\mathcal{V}(3,0)$. In that case we can use $I \in \mathcal{G}(5,1)$ and the dual of the grade 2 blade in equation 81 with be of grade 4 which is correct for a circle or line.

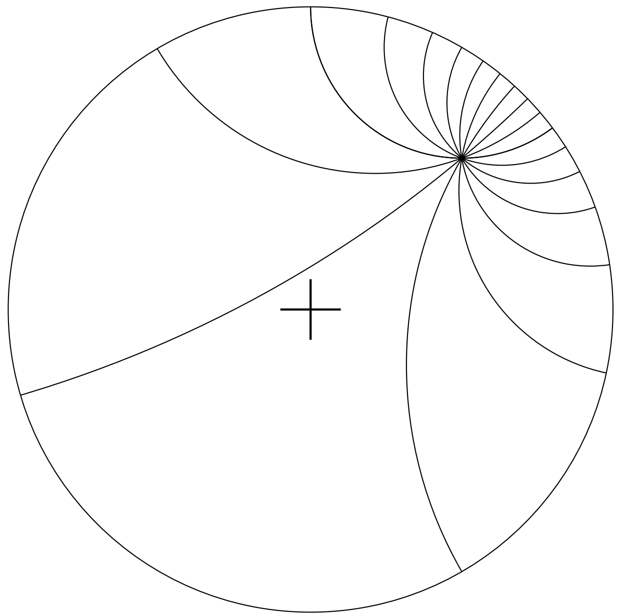
Non-Euclidian Geometry

Start by looking at the properties of the blade that defines a line, $L = X \wedge Y \wedge n$, in Euclidian space. The generator of the line is (note that $(X \cdot n) = (Y \cdot n) = -1$)

$$
\begin{aligned}
B = Ln = (X \wedge Y \wedge n)\, n &= ((Y \cdot n)\, X - (X \cdot n)\, Y) \wedge n && (82)\\
&= (Y - X) \wedge n && (83)\\
&= (y - x) \wedge n && (84)\\
&= (y - x)\, n && (85)
\end{aligned}
$$

Thus $R = \mathrm{e}^{\frac{B}{2}}$ is the translation that takes x into y and $R = \mathrm{e}^{\frac{\alpha}{2}B}$ is a translation that would translate points on the line defined by x and y along the line so that we call B the generator of the line. Another critical property of R is that $RnR^\dagger = n$. The point at infinity is not affected by a translation operation.

Our introduction to non-euclidian geometry is to consider the consequences of replacing n by e.



Straight Lines In the Non-Euclidian Plane $L = X \wedge Y \wedge e$

Thus the equivalent of a straight line in our new geometry is defined by $L \wedge Z = 0$ where $L = X \wedge Y \wedge e$. In two dimensions $(F(x) \wedge F(y) \wedge e) \wedge F(z) = 0$ is equivalent to[2]

$$\begin{gathered} D\left(1+z^2\right) + Az_0 + Bz_1 = 0 \\ D = x_0 y_1 - x_1 y_0 \\ A = x_1\left(1+y^2\right) - y_1\left(1+x^2\right) \\ B = y_0\left(1+x^2\right) - x_0\left(1+y^2\right) \end{gathered} \tag{86}$$

or

$$\left(z_0 + \frac{A}{2D}\right)^2 + \left(z_1 + \frac{B}{2D}\right)^2 = \frac{A^2+B^2}{4D^2} - 1 \tag{87}$$

If $D = 0$ we have a straight line and if $\frac{A^2+B^2}{4D^2} - 1 > 0$ we have a circle. After much work the condition for a circle can be reduced to

$$\left(1-x^2\right)\left(1-y^2\right) > 0 \tag{88}$$

[2]Equations 86, 92, 97, 98, and 99 were calculated with GAsympy.py.

so that L is a circle if $x^2, y^2 < 1$.

For the generator of the line we have in analogy to Euclidian space:

$$B = Le = (X \wedge Y \wedge e)\, e = X \wedge Y + (-(Y \cdot e))\, X \wedge e + ((X \cdot e))\, Y \wedge e \tag{89}$$

Thus B is a generator of rotations in the conformal space. Next the equivalent normalization of X is $X \cdot e = -1$ (instead of $X \cdot n = -1$) giving

$$X = F(x) = \frac{1}{1 - x^2}\left(x^2 n + 2x - \bar{n}\right) \tag{90}$$

and

$$X \cdot Y = \frac{-2\,(x - y)^2}{(1 - x^2)\,(1 - y^2)} \tag{91}$$

$$B^2 = (X \cdot Y)^2 - 2\,(X \cdot Y)\,(X \cdot e)\,(Y \cdot e) \tag{92}$$

Thus $B^2 > 0$ forall $x^2 < 1$ and $y^2 < 1$ or $x^2 > 1$ and $y^2 > 1$. So that if we define $|B| = \sqrt{B^2}$ and $\hat{B} = \dfrac{B}{|B|}$

$$R = \mathsf{e}^{\frac{\alpha}{2}\hat{B}} = \cosh\left(\frac{\alpha}{2}\right) + \sinh\left(\frac{\alpha}{2}\right)\hat{B} \tag{93}$$

and

$$\begin{aligned} ReR^{\dagger} &= \left(\cosh\left(\frac{\alpha}{2}\right)^2 - \frac{\sinh\left(\frac{\alpha}{2}\right)^2}{B^2}\left((X\cdot Y)^2 - 2\,(X\cdot Y)\,(X\cdot e)\,(Y\cdot e)\right)\right)e \\ &= \left(\cosh\left(\frac{\alpha}{2}\right)^2 - \sinh\left(\frac{\alpha}{2}\right)^2\right)e \\ &= e \end{aligned} \tag{94}$$

So that e is invarient under rotations generated by R. The next step is to determine what value of α will cause X to be translated to Y or

$$Y = RXR^{\dagger} \tag{95}$$

Since Y is null the equation to solve is

$$Y^2 = Y \cdot Y = \left(RXR^{\dagger}\right) \cdot Y = 0 \tag{96}$$

This is equivalent to

$$\cosh(\alpha)\left(\left(X \cdot Y\right) - \left(X \cdot e\right)\left(Y \cdot e\right)\right) + \left(X \cdot e\right)\left(Y \cdot e\right) + \sinh(\alpha)\left|B\right| = 0 \tag{97}$$

or

$$\cosh(\alpha) = 1 - \frac{(X \cdot Y)}{(X \cdot e)(Y \cdot e)} \tag{98}$$

or

$$\sinh^2\left(\frac{\alpha}{2}\right) = -\frac{X \cdot Y}{2(X \cdot e)(Y \cdot e)} \tag{99}$$

so

$$
\begin{aligned}
d\left(x,y\right) &= 2\sinh^{-1}\left(\sqrt{-\frac{X\cdot Y}{2\left(X\cdot e\right)\left(Y\cdot e\right)}}\right) \\
&= 2\sinh^{-1}\left(\sqrt{\frac{\left(x-y\right)^2}{\left(1-x^2\right)\left(1-y^2\right)}}\right) \qquad (100)
\end{aligned}
$$

If x, y, and z are all on the same non-euclidian line then

$$
d\left(x,y\right)+d\left(y,z\right)=d\left(x,z\right) \qquad (101)
$$